NPS71-83-002

NAVAL POSTGRADUATE SCHOOL
Monterey, California

PROGRAMS FOR A TARGET
POSITION ESTIMATION PROCEDURE

BY

R. N. FORREST

March 1983

Approved for public release; distribution unlimited.

Prepared for: Naval Postgraduate School
Monterey, CA 93940

NAVAL POSTGRADUATE SCHOOL
Monterey, California

Rear Admiral J. J. Ekelund
Superintendent

David A. Schrady
Provost

Reproduction of all or part of this report is authorized.

This report prepared by:

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (*When Data Entered*)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NPS71-83-002	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (*and Subtitle*) Programs for a Target Position Estimation Procedure		5. TYPE OF REPORT & PERIOD COVERED Technical 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) R. N. Forrest		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, CA 93940		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Postgraduate School Monterey, Ca 93940		12. REPORT DATE March 1983 13. NUMBER OF PAGES 48
14. MONITORING AGENCY NAME & ADDRESS (*if different from Controlling Office*)		15. SECURITY CLASS. (*of this report*) Unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (*of this Report*)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (*of the abstract entered in Block 20, if different from Report*)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (*Continue on reverse side if necessary and identify by block number*)

Bearings-only Position Estimation
Calculator Programs
Programmable Calculators
Probability Areas

20. ABSTRACT (*Continue on reverse side if necessary and identify by block number*)

The report contains program listings and user instructions for an HP-41CV, a Sharp PC-1500 (or TRS-80 PC-2), a Sharp PC-1211 (or TRS-80 PC-1), a Casio FX-702P and a TI-59. The programs implement a bearings-only position estimation procedure. A development for the procedure is included in the report.

DD FORM 1473, 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-LF-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (*When Data Entered*)

The programs in this report are for use within the Department of the Navy, and they are presented without representation or warranty of any kind.

TABLE OF CONTENTS

I. Introduction

This report contains a target position estimation program for each of the following calculators: The Hewlett-Packard HP-41CV, the Sharp PC-1500 (or Radio Shack TRS-80 PC-2), the Sharp PC-1211 (or Radio Shack TRS-80 PC-1), the Casio FX-702P and the Texas Instruments TI-59.

The programs provide a means of implementing a position estimation procedure that is described in Appendix 1. The procedure is based on the following assumptions: Bearings taken on or from a target are available from two or more stations of known position. The positions of the stations and the target are such that they can be considered to be on the surface of a plane (a flat earth model). The error in a bearing taken on or from a station is a normal random variable. Its standard deviation (bearing error) is known and its mean (bias) is zero (if there is bias in a bearing, it is known and it is removed). The errors in bearing measurements are independent.

The position estimation procedure requires an initial estimate of the target's position. In the programs that are given here, the initial estimate is at the intersection of the bearing lines determined by the first two target bearings that are input to the program. Because of the use of this method to choose an initial estimate, the first two stations, with respect to the order of data input, should be chosen so that the intersection of their bearing lines is likely to be closer to the target's position than the intersection of the bearing lines from any other pair of stations.

The target ranges and errors of the bearings of the first two stations determine the distance between the intersection of their bearing lines and the target's position. In particular, if the angular separation between two stations as seen from the target is small relative to the bearing error of one or both of the stations, the bearing lines from the two stations may not intersect. If they do not intersect and they are not parallel, their reciprocal bearing lines will. In this case, if the observed bearings from the two stations were the first two bearing inputs to the program, the initial estimate of the target's position would be at the intersection of the two reciprocal bearing lines, and a gross error in the final position estimate could result.

II. Program User Instructions

Before using a program for the first time, refer to the notes that follow the user's instructions. Also, note the comments in Section I regarding the relationship between the accuracy of the computed estimates and the order of data entry.

For a PC-1211, use the PC-1500 User Instructions. Use the DEF mode and substitute SHFT for DEF wherever DEF appears in the instructions.

Station positions are determined with respect to a known reference point. The reference point can be a station position, in this case the station bearing and range from the reference point are both zero.

Program listings are given in Section IV and two example applications are discussed in Section III.

Position Estimation Program

HP-41CV User Instructions

Step	Instruction	Prompt	Press
1	Select USER mode and enter the program (see Note 1). Press Σ+ to run the program.		Σ+
2	Key in, in decimal degrees, the observed bearing of the target from a station or the reciprocal of the observed bearing of the station from the target (see Note 2).	OBS BRG?	R/S
3	Key in the station bearing in decimal degrees from the reference point. Use zero if the station is the reference point.	STA BRG?	R/S
4	Key in the station range (in any units) from the reference point. Use zero if the station is the reference point.	STA RNG?	R/S
5	Key in the bearing error (standard deviation) in decimal degrees.		R/S
6	Repeat Steps 2, 3, 4 & 5 for one or more additional stations.		
7	Compute bearing & range estimates.		1/x
8	A computed target bearing estimate in decimal degrees is displayed.	BRG=est.	R/S
9	A computed target range estimate in meters. For an elliptical containment region, go to Step 10 or Step 17. To enter additional data from new or old stations, go to Step 24.	RNG=est.	
10	For a containment ellipse of a given containment probability, press LN.		LN
11	Key in the desired containment probability.	PRB?	R/S
12	Computed value of the ellipse size: (See Note 3.)	SIZE=val.	R/S

Step	Instruction	Prompt	Press
13	Computed semi-major axis length:	SMJ=val.	R/S
14	Computed major axis direction:	DRC=val.	R/S
15	Computed semi-minor axis length:	SMI=val.	R/S
16	Computed containment ellipse area: (See Note 4.)	A = val.	
17	For a containment ellipse of a given size, press LOG.		LOG
18	Key in the desired containment ellipse size (see Note 3).	SIZE?	R/S
19	Computed value of the containment probability:	PRB=val.	R/S
20	Computed semi-major axis length:	SMJ=val.	R/S
21	Computed major axis direction:	DRC=val.	R/S
22	Computed semi-minor axis length:	SMI=val.	R/S
23	Computed containment ellipse area: (See Note 4.)	A = val.	
24	To enter additional data from new or old stations, press $\sqrt{x}$ and then repeat Steps 2, 3, 4 & 5		$\sqrt{x}$

Notes:

1. The program size is 35. The key assignments are: TPE → Σ+, CON → √x, EST → 1/x, SIZ → LOG and PRB → LN. If they are not present, they must be made in order to follow the user instructions. For an alternative, see Appendix 2.
2. Reciprocal bearings are used when bearings are taken on known positions from the unknown position (target).
3. In the model that is the basis for the program, for a given probability of containment, the minimum area containment region is an ellipse centered on the position estimate. The semi-major axis = $k\sigma_{MJ}$ and semi-minor axis = $k\sigma_{MI}$ where k is the size of the ellipse and σ_{MJ} and σ_{MI} are the standard deviations (uncertainty measure) of the position estimate in the major axis and minor axis directions.
4. The area units are the range units squared.

Position Estimation Program

PC-1500 User Instructions

Step	Instruction	Prompt	Press
1	Enter the program. To run the program, press DEF, A. For a PC-1211, see Note 5.		DEF A
2	Key in, in decimal degrees, the observed bearing of the target from a station or the reciprocal of the observed bearing of the station from the target (see Note 1).	OBS BRG?	ENTER
3	Key in the station bearing in decimal degrees from the reference point. Use zero if the station is the reference point.	STA BRG?	ENTER
4	Key in the station range (in any units) from the reference point. Use zero if the station is the reference point.	STA RNG?	ENTER
5	Key in the bearing error (standard deviation) in decimal degrees.	BRG ERR?	ENTER
6	Repeat Steps 2, 3, 4 & 5 for one or more additional stations.		
7	Compute bearing & range estimates.		DEF Z
8	A computed target bearing estimate in decimal degrees is displayed.	BRG=est.	ENTER
9	A computed target range estimate is displayed. For an elliptical containment region, go to Step 10 or Step 17. To enter additional data from new or old stations, go to Step 24.	RNG=est.	
10	For a containment ellipse of a given containment probability, press DEF, X.		DEF X
11	Key in the desired containment probability.	PRB?	ENTER
12	Computed value of the ellipse size: See Note 2.	SIZE=val.	ENTER

Step	Instruction	Prompt	Press
13	Computed semi-major axis length:	SMJ=val.	ENTER
14	Computed major axis direction:	DRC=val.	ENTER
15	Computed semi-minor axis length:	SMI=val.	ENTER
16	Computed containment ellipse area: (See Note 3.)	A = val.	
17	For a containment ellipse of a given size, press DEF, S.		DEF S
18	Key in the desired containment ellipse size (see Note 2).	SIZE?	ENTER
19	Computed value of the containment probability:	PRB=val.	ENTER
20	Computed semi-major axis length:	SMJ=val.	ENTER
21	Computed major axis direction:	DRC=val.	ENTER
22	Computed semi-minor axis length:	SMI=val.	ENTER
23	Computed containment ellipse area: (See Note 3.)	A = val.	
24	To enter additional data from new or old stations, press DEF, C and then repeat Steps 2, 3, 4 & 5.		DEF C

Notes:

1. Reciprocal bearings are used when bearings are taken on known positions from the unknown position (target).
2. In the model that is the basis for the program, for a given probability of containment, the minimum area containment region is an ellipse centered on the position estimate. The semi-major axis = $k\sigma_{MJ}$ and semi-minor axis = $k\sigma_{MI}$ where k is the size of the ellipse and σ_{MJ} and σ_{MI} are the standard deviations (uncertainty measure) of the position estimate in the major axis and minor axis directions.
3. The area units are the range units squared.
4. For a definition of the program initiating keys and their function, press DEF, H. The display will show: TPE = A EST = Z SIZ = S PRB = X. Next press ENTER. The display will show: CON = C. To repeat the display, press ENTER.
5. For a PC-1211, use the DEF mode and substitute SHFT for DEF wherever DEF appears in the instructions.

Position Estimation Program

FX-702P User Instructions

Step	Instruction	Prompt	Press
1	Enter the program (see Note 1). To run the program, first press F1, Ø.		F1 Ø
2	Next, key in 1 and press EXE.	OPTION?	1 EXE
3	Key in, in decimal degress, the observed bearing of the target from a station or the reciprocal of the observed bearing of the station from the target (see Note 2).	OBS BRG?	EXE
4	Key in the station bearing in decimal degrees from the reference point. Use zero if the station is the reference point.	STA BRG?	EXE
5	Key in the station range (in any units) from the reference point. Use zero if the station is the reference point.	STA RNG?	EXE
6	Key in the bearing error (standard deviation) in decimal degrees.	BRG ERR?	EXE
7	Repeat Steps 3, 4, 5 & 6 for one or more additional stations.		
8	To compute bearing & range estimates, first press F1, Ø.		F1 Ø
9	Next, key in 2 and press EXE.	OPTION?	2 EXE
10	A computed target bearing estimate in decimal degrees is displayed.	BRG=est.	CONT
11	A computed target range estimate is displayed. For an elliptical containment region, go to Step 12 or Step 20. To enter additional data from new or old stations, go to Step 28.	RNG=est.	
12	For a containment ellipse of a given containment probability, first press F1, Ø.		F1 Ø
13	Next, key in 3 and press EXE.	OPTION?	3 EXE

Step	Instruction	Prompt	Press
14	Key in the desired probability.	PRB?	EXE
15	Computed value of the ellipse size: See Note 3.	SIZE=val.	CONT
16	Computed semi-major axis length:	SMJ=val.	CONT
17	Computed major axis direction:	DRC-val.	CONT
18	Computed semi-minor axis length:	SMI=val.	CONT
19	Computed containment ellipse area: See Note 4.	A = val.	
20	For a containment ellipse of a given size, first press F1, Ø.		F1 Ø
21	Next, key in 4 and press EXE.	OPTION?	4 EXE
22	Key in the desired containment ellipse size (see Note 3).	SIZE?	EXE
23	Computed value of the containment probability:	PRB=val.	CONT
24	Computed semi-major axis length:	SMJ=val.	CONT
25	Computed major axis direction:	DRC=val.	CONT
26	Computed semi-minor axis length:	SMI=val.	CONT
27	Computed containment ellipse area: See Note 4.	A = val.	
28	To enter additional data from new or old stations, first press F1, Ø.		F1 Ø
29	Next, key in 5, press EXE and then repeat Steps 3, 4, 5 & 6.	OPTION?	5 EXE

Notes:

1. Enter the program in PØ for F1 Ø activation. Before running the program, first press F2 then DEFM then 1 and then EXE. This is required in order to use the array variables in the program.
2. Reciprocal bearings are used when bearings are taken on known positions from the unknown position (target).
3. In the model that is the basis for the program, for a given probability of containment, the minimum area containment region is an ellipse centered on the position estimate. The semi-major axis = $k\sigma_{MJ}$ and semi-minor axis = $k\sigma_{MI}$ where k is the size of the ellipse and σ_{MJ} and σ_{MI} are the standard deviations (uncertainty measure) of the position estimate in the major axis and minor axis directions.
4. The area units are the range units squared.
5. For a definition of the program options and their function, press F1, Ø. Then after OPTION? is displayed, enter Ø and press EXE. The display will show: TPE = 1 EST = 2 SIZ = 3. Next, press CONT. The display will show: PRB = 4 CON = 5. To repeat the displays, press CONT.

POSITION ESTIMATION PROGRAM

TI-59 USER INSTRUCTIONS

Step	Instruction	Enter	Press	Display
1	Enter the program (see Note 1).			
2	To run the program, press A.		A	9
3	Enter, in decimal degrees, the observed bearing of the target from a station or the reciprocal of the observed bearing of the station from the target (see Note 2).	θ	B	θ
4	Enter the station bearing in decimal degrees from the reference point. Use zero if the station is the reference point.	α	R/S	α
5	Enter the station range from the reference point. Use zero if the station is the reference point. Use any units.	ρ	R/S	ρ
6	Enter the bearing error (standard deviation) in decimal degrees. After pressing R/S, the display indicates the order number of the data entry.	e	R/S	n
7	Repeat Steps 3, 4, 5 & 6 for at least one more station.			
8	Display a bearing estimate $\hat{\phi}$ with respect to the reference station.		C	$\hat{\phi}$
9	Display a range estimate $\hat{r}$ with respect to the reference station. For an elliptical containment region, go to Step 10 or Step 15. To enter additional data from new or old stations, repeat Steps 3, 4, 5 & 6.		R/S	$\hat{r}$
10	For a containment ellipse of a given containment probability, enter the containment probability. Next, press E and display the ellipse size (see Note 3).	p	E	k
11	Display the semi-major axis length.		R/S	$k\sigma_{MJ}$
12	Display the major axis direction.		R/S	γ
13	Display the semi-minor axis length.		R/S	$k\sigma_{MI}$

Step	Instruction	Enter	Press	Displ
14	Display the containment ellipse area. (See Note 4.)		R/S	area
15	For a containment ellipse of a given size, enter the containment ellipse size (See Note 3). Next, press D and display the containment probability.	k	D	p
16	Display the semi-major axis length.		R/S	$k\sigma_{MJ}$
17	Display the major axis direction.		R/S	γ
18	Display the semi-minor axis length.		R/S	$k\sigma_{MI}$
19	Display the containment ellipse area. (See Note 4.)		R/S	area
20	To enter additional data from new or old stations, repeat Steps 3, 4, 5 & 6.			

Notes:

1. The program requires the normal partition. If the calculator has been in use, this can be assured by turning the calculator off and then on before loading the program.
2. Reciprocal bearings are used when bearings are taken on known positions from the unknown position (target).
3. In the model that is the basis for the program, for a given probability of containment, the minimum area containment region is an ellipse centered on the position estimate. The semi-major axis = $k\sigma_{MJ}$ and semi-minor axis = $k\sigma_{MI}$ where k is the size of the ellipse and σ_{MJ} and σ_{MI} are the standard deviations (uncertainty measure) of the position estimate in the major axis and minor axis directions.
4. The area units are the range units squared.
5. Negative bearing estimates and negative major axis directions can result. To convert a negative bearing estimate to the value that would be output by the other programs, add 360° to the estimate. For example, -5° becomes 355°. To convert a negative direction, add 180°. For example, -5° becomes 175°.

III. Two Examples

In Scenario 1, the scenario for the first example, bearings are taken on a target from three separate stations. Figure 1 on Page 18 shows Scenario 1 and Table 1 below gives the station data. The stations are numbered according to the order of station data input to the program.

TABLE 1

	OBS BRG	STA BRG	STA RNG	BRG ERR
Station 1	038°	334°	13500	4°
Station 2	324°	050°	11350	3°
Station 3	003°	000°	0	4°

Note from Table 1 that the reference point is at Station 3.

Program outputs for Scenario 1 are given in List 1 on Page 19. List 1 and List 2 which gives program outputs for Scenario 2, are copies of printer tapes that were generated with a Casio FP-10 printer and a Casio FX-702P using the Casio FX-702P program. With allowance for differences in user instructions, display formats and round-off errors, the tapes indicate the output that should be obtained using any of the other calculator programs except for the TI-59 program where equivalent negative angles occur.

In Scenario 2, the scenario for the second example, bearings are taken from a target on three separate stations. Figure 2 on Page 20 shows Scenario 2 and Table 2 below gives the station data. As in Scenario 1, the stations are numbered according to the order of station data input to the program. In this scenario the reciprocal of the observed bearings are used in order to provide an equvalent scenario that is appropriate for the program.

TABLE 2

	OBS BRG	RCP BRG	STA BRG	STA RNG	BRG ERR
Station 1	211°	031°	000°	0	3°
Station 2	172°	352°	115°	11100	3°
Station 3	146°	326°	082°	13800	3°

Note from Table 2 that the reference point is at Station 1.

Program outputs for Scenario 2 are given in List 2 on Page 21. The two bearing estimates in the list are the reciprocal of the target's estimate of the bearing of Station 1.

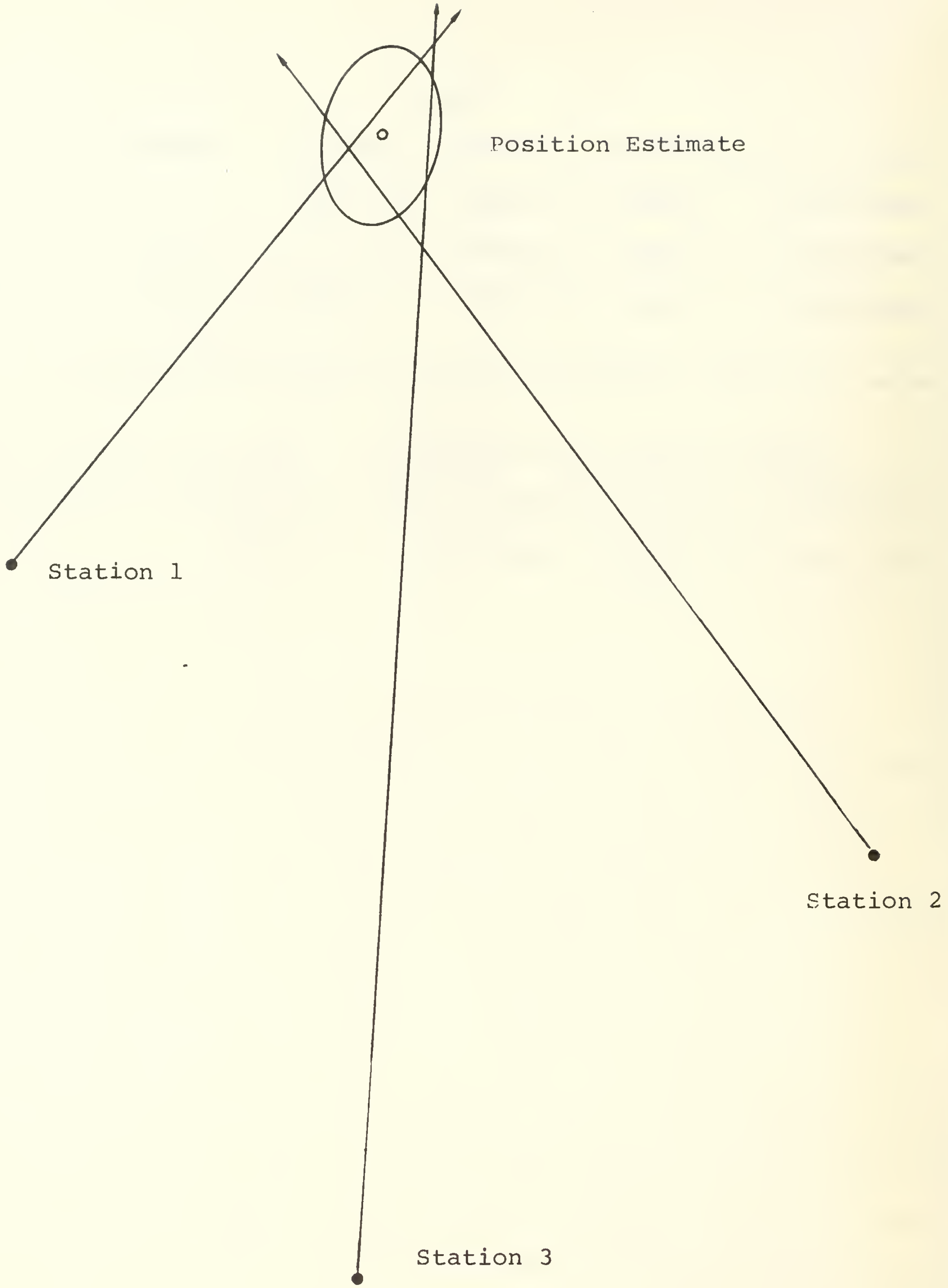

Figure 1. A position estimate and a .33 confidence region. Table 1 gives the bearings of the indicated bearing lines and List 1 gives the bearing and the range of the position estimate from Station 3.

```
OPTION?
0
TPE=1 EST=2 SIZ=3

PRB=4 CON=5

OPTION?
1
OBS BRG?
38
STA BRG?
334
STA RNG?
13500
BRG ERR?
4
OBS BRG?
324
STA BRG?
50
STA RNG?
11350
BRG ERR?
3
OBS BRG?

OPTION?
2
BRG= 359.51

RNG= 19494.39

END

OPTION?
3
SIZE?
2
PRB= 0.86

SMJ= 1737.32

DIR= 17.69

SMI= 1232.96

A= 6729444.91

END

OPTION?
4
PRB?
.9
SIZE= 2.15

SMJ= 1964.12

DIR= 17.69

SMI= 1322.94

A= 7747559.77

END
```

```
OPTION?
5
OBS BRG?
3
STA BRG?
0
STA RNG?
0
BRG ERR?
4
OBS BRG?

OPTION?
2
BRG= 0.15

RNG= 19553.76

END

OPTION?
3
SIZE?
2
PRB= 0.86

SMJ= 1712.95

DIR= 12.48

SMI= 1129.90

A= 6080418.40

END

OPTION?
4
PRB?
.9
SIZE= 2.15

SMJ= 1837.97

DIR= 12.48

SMI= 1212.36

A= 7000340.38

END
```

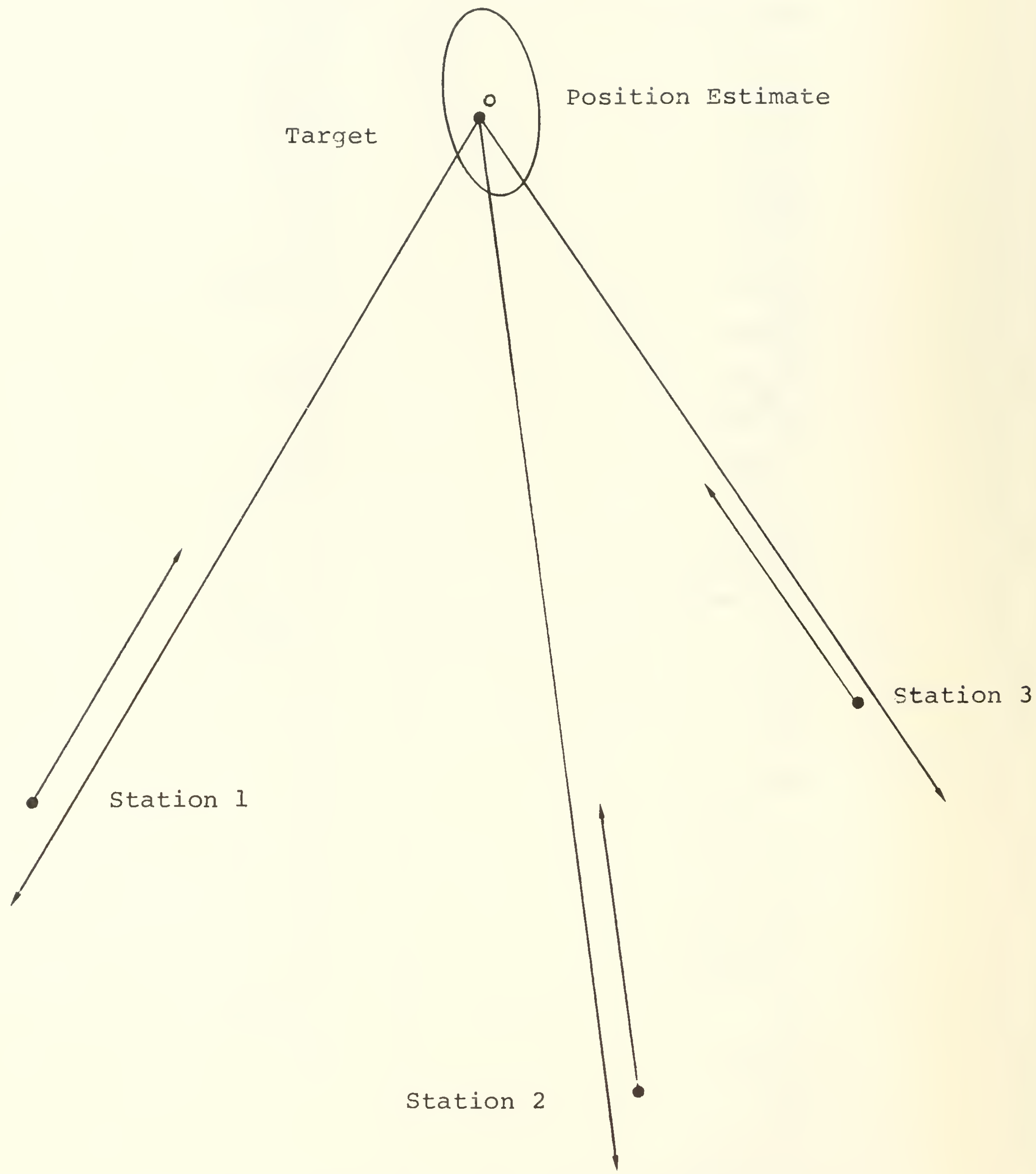

Figure 2. A position estimate and .71 confidence region. Table 2 gives the bearings of the indicated bearing lines and List 2 gives the reciprocal bearing and the range of the position estimate from Station 1.

```
OPTION?
0
TPE=1 EST=2 SIZ=3

PRB=4 CON=5

OPTION?
1
OBS BRG?
31
STA BRG?
0
STA RNG?
0
BRG ERR?
3
OBS BRG?
352
STA BRG?
115
STA RNG?
11100
BRG ERR?
3
OBS BRG?

OPTION?
2
BRG= 31.00

RNG= 14792.53

END

OPTION?
3
SIZE?
2
PRB= 0.86

SMJ= 3606.56

DIR= 15.39

SMI= 1253.72

A= 14205070.91

END

OPTION?
4
PRB?
.9
SIZE= 2.15

SMJ= 3869.77

DIR= 15.39

SMI= 1345.22

A= 16354192.26

END

OPTION?
5
OBS BRG?
326
STA BRG?
82
STA RNG?
13800
BRG ERR?
3
OBS BRG?

OPTION?
2
BRG= 31.65

RNG= 13765.30

END

OPTION?
3
SIZE?
2
PRB= 0.86

SMJ= 2023.19

DIR= 172.06

SMI= 967.90

A= 6151983.32

END

OPTION?
4
PRB?
.9
SIZE= 2.15

SMJ= 2170.85

DIR= 172.06

SMI= 1038.54

A= 7082732.54

END
```

IV. Program Listings

HP-41CV PROGRAM

```
01♦LBL "TPE"
02 DEG
03 CLRG
04 .001
05 STO 09
06 26
07 STO 10
08♦LBL "CON"
09 FIX 0
10 "OBS BRG?"
11 PROMPT
12 STO 11
13 "STA BRG?"
14 PROMPT
15 STO 12
16 "STA RNG?"
17 PROMPT
18 STO 13
19 "BRG ERR?"
20 PROMPT
21 STO 14
22 1
23 ST+ 08
24 2
25 RCL 08
26 X>Y?
27 GTO 02
28 1
29 ST+ 10
30 RCL 11
31 STO IND 10
32 1
33 ST+ 10
34 RCL 12
35 STO IND 10
36 1
37 ST+ 10
38 RCL 13
39 STO IND 10
40 1
41 ST+ 10
42 RCL 14
43 STO IND 10
44 1
45 RCL 08
46 X=Y?
47 GTO "CON"
48 RCL 29
49 RCL 28
50 RCL 27
51 -
52 SIN
53 *
54 STO 15
55 RCL 33
56 RCL 32
57 RCL 31
58 -
59 SIN
60 *
61 STO 16
62 RCL 31
63 RCL 27
64 -
65 SIN
66 STO 17
67 X=0?
68 GTO 08
69 RCL 15
70 RCL 31
71 SIN
72 *
73 RCL 16
74 RCL 27
75 SIN
76 *
77 -
78 RCL 17
79 /
80 STO 01
81 RCL 15
82 RCL 31
83 COS
84 *
85 RCL 16
86 RCL 27
87 COS
88 *
89 -
90 RCL 17
91 /
92 STO 02
93 26
94 STO 10
95♦LBL 05
96 1
97 ST+ 10
98 RCL IND 10
99 STO 11
100 1
101 ST+ 10
102 RCL IND 10
103 STO 12
104 1
105 ST+ 10
106 RCL IND 10
107 STO 13
108 1
109 ST+ 10
110 RCL IND 10
111 STO 14
112 XEQ 07
113 ISG 09
114 GTO 05
115 GTO "CON"
116♦LBL 02
117 XEQ 07
118 GTO "CON"
119♦LBL 07
120 RCL 01
121 RCL 13
122 RCL 12
123 SIN
124 *
125 -
126 RCL 02
127 RCL 13
128 RCL 12
129 COS
130 *
131 -
132 R-P
133 STO 16
134 X<>Y
135 STO 15
136 RCL 11
137 RCL 15
138 -
139 180
140 X<>Y
141 X<=Y?
142 GTO 11
143 360
144 -
145♦LBL 11
146 RCL 14
147 /
148 STO 17
149 RCL 16
150 RCL 14
```

```
*
PI
*
180
/
STO 18
RCL 15
COS
RCL 18
/
STO 19
RCL 15
SIN
RCL 18
/
STO 20
RCL 19
X↑2
ST+ 03
RCL 19
RCL 20
*
ST+ 04
RCL 20
X↑2
ST+ 05
RCL 17
RCL 19
*
ST+ 06
RCL 17
RCL 20
*
ST+ 07
RTN
♦LBL "EST"
FIX 2
RCL 04
X↑2
RCL 03
RCL 05
*
-
STO 20
X=0?
GTO 08
RCL 01
RCL 04
RCL 07
*
RCL 05
RCL 06
*
-
RCL 20
/
+
STO 26
RCL 02
RCL 03
RCL 07
*
RCL 04
RCL 06
*
-
RCL 20
/
+
RCL 26
X<>Y
R-P
STO 21
X<>Y
X>0?
GTO 06
360
+
♦LBL 06
STO 22
RCL 04
RCL 03
RCL 05
X≠Y?
GTO 03
RCL 04
SIGN
45
*
GTO 04
♦LBL 03
-
/
2
*
ATAN
2
/
♦LBL 04
STO 23
COS
X↑2
RCL 05
*
RCL 23
1
P-R
*
2
*
RCL 04
*
STO 26
-
RCL 23
SIN
X↑2
RCL 03
*
+
RCL 20
CHS
/
STO 24
RCL 23
SIN
X↑2
RCL 05
*
RCL 26
+
RCL 23
COS
X↑2
RCL 03
*
+
RCL 20
CHS
/
STO 25
RCL 24
X<=Y?
GTO 01
STO 25
X<>Y
STO 24
90
ST+ 23
♦LBL 01
```

```
301 "BRG="
302 ARCL 22
303 AVIEW
304 STOP
305 "RNG="
306 ARCL 21
307 AVIEW
308 STOP
309 "END"
310 GTO 00
311♦LBL "SIZ"
312 "SIZE"
313 PROMPT
314 STO 11
315 X↑2
316 2
317 /
318 CHS
319 E↑X
320 1
321 X<>Y
322 -
323 STO 00
324 "PRB="
325 ARCL 00
326 AVIEW
327 STOP
328 GTO 09
329♦LBL "PRB"
330 "PRB?"
331 PROMPT
332 CHS
333 1
334 +
335 LN
336 2
337 *
338 CHS
339 SQRT
340 STO 11
341 "SIZE="
342 ARCL 11
343 AVIEW
344 STOP
345♦LBL 09
346 RCL 23
347 STO 12
348 RCL 11
349 RCL 25
350 SQRT
351 *
352 RCL 11
353 RCL 24
354 SQRT
355 *
356 X<>Y
357 "SMJ="
358 ARCL X
359 AVIEW
360 STOP
361 RCL 23
362 X>0?
363 GTO 10
364 180
365 +
366♦LBL 10
367 "DRC="
368 ARCL X
369 AVIEW
370 STOP
371 "SMI="
372 RDN
373 ARCL Y
374 AVIEW
375 STOP
376 PI
377 *
378 *
379 "A="
380 ARCL X
381♦LBL 00
382 AVIEW
383 STOP
384 "END"
385 GTO 00
386♦LBL 08
387 AVIEW
388 STOP
389 "NO SOL"
390 GTO 08
391 END
```

```
10:"A":CLEAR :DIM
   A(7):DEGREE
15:"C":INPUT "OBS
    BRG? ";P:
   INPUT "STA BRG
   ? ";Q:INPUT "S
   TA RNG? ";R:
   INPUT "BRG ERR
   ? ";O:PAUSE "
   "
20:IF I=2GOTO 50
25:I=I+1:A(I-1)=P
   :A(I+1)=Q:A(I+
   3)=R:A(I+5)=O:
   IF I=1GOTO 15
30:X=A(4)*SIN (A(
   2)-A(0)):Y=A(5
   )*SIN (A(3)-A(
   1)):Z=SIN (A(1
   )-A(0)):IF Z=0
   GOTO 150
35:U=(X*SIN A(1)-
   Y*SIN A(0))/Z:
   V=(X*COS A(1)-
   Y*COS A(0))/Z
40:FOR M=0TO 1
45:P=A(M):Q=A(M+2
   ):R=A(M+4):O=A
   (M+6):GOSUB 17
   0:NEXT M:GOTO
   15
50:GOSUB 170:GOTO
   15
60:"Z":PAUSE " ":
   F=(B*B-A*C):IF
   F=0GOTO 150
65:X=U+(B*E-C*D)/
   F:Y=V+(A*E-B*D
   )/F:GOSUB 200
70:T=SGN B*45:IF
   A=CGOTO 80
75:T=.5*ATN (2*B/
   (A-C))
80:G=(C*COS T*COS
   T-2*B*COS T*
   SIN T+A*SIN T*
   SIN T)/-F:G=√G
85:H=(C*SIN T*SIN
   T+2*B*COS T*
   SIN T+A*COS T*
   COS T)/-F:H=√H
   :IF H>=GGOTO 9
   5
90:Z=H:H=G:G=Z:T=
   T+90
95:PRINT "BRG=";J
   :PRINT "RNG=";
   K:GOTO 145
100:"H":PRINT " TP
    E=A EST=Z SIZ=
    S PRB=X"
105:PRINT "CON=C":
    GOTO 100
120:"S":INPUT "SIZ
    E? ";S:O=1-EXP
    (-S*S/2)
125:PRINT USING "#
    #.##";"PRB=";O
    :USING :GOTO 1
    35
130:"X":INPUT "PRB
    ? ";O:S=√(-2*
    LN (1-O)):
    PRINT "SIZE=";
    S
135:X=S*G:Y=S*H:N=
    T:PRINT "SMJ="
    ;Y:IF N<0LET N
    =N+180
140:PRINT "DIR=";N
    :PRINT "SMI=";
    X:PRINT "A=";π
    *X*Y
145:PRINT "END":
    GOTO 145
150:PRINT "NO SOL"
    :GOTO 150
170:X=U-R*SIN Q:Y=
    V-R*COS Q:
    GOSUB 200
175:W=(P-J):L=K*O*
    π/180:G=COS J/
    L:H=SIN J/L:IF
    W>=180LET W=W-
    360:GOTO 185
180:IF W<=-180LET
    W=W+360
185:W=W/O:A=G*G+A:
    B=G*H+B:C=H*H+
    C:D=W*G+D:E=W*
    H+E:RETURN
200:K=√(X*X+Y*Y):
    IF K=0LET J=0:
    RETURN
205:J=ACS (Y/K):IF
    ASN (X/K)<0LET
    J=360-J
210:RETURN
```

```
 10:"A":CLEAR :
    DEGREE
 15:"C":INPUT "O
    BS BRG? ";P:
    INPUT "STA B
    RG? ";Q:
    INPUT "STA R
    NG? ";R:
    INPUT "BRG E
    RR? ";O:IF I
    =2GOTO 130
 75:I=I+1:A(I+26
    )=P:A(I+28)=
    Q:A(I+30)=R:
    A(I+32)=O:IF
    I=1GOTO 15
 85:X=A(31)*SIN
    (A(29)-A(27)
    ):Y=A(32)*
    SIN (A(30)-A
    (28)):Z=SIN
    (A(28)-A(27)
    )
 90:IF Z=0GOTO 3
    25
 95:U=(X*SIN A(2
    8)-Y*SIN A(2
    7))/Z:V=(X*
    COS A(28)-Y*
    COS A(27))/Z
105:FOR I=1TO 2:
    P=A(I+26):Q=
    A(I+28):R=A(
    I+30):O=A(I+
    32):GOSUB 40
    0
110:NEXT I:GOTO
    15
130:GOSUB 400
135:GOTO 15
140:"Z":F=(B*B-A
    *C):IF F=0
    GOTO 325
150:X=U+(B*E-C*D
    )/F:Y=V+(A*E
    -B*D)/F:
    GOSUB 500
160:T=SGN B*45:
    IF A=CTHEN 1
    70
165:T=.5*ATN (2*
    B/(A-C))
170:G=(C*COS T*
    COS T-2*B*
    COS T*SIN T+
    A*SIN T*SIN
    T)/-F:G=√G
175:H=(C*SIN T*
    SIN T+2*B*
    COS T*SIN T+
    A*COS T*COS
    T)/-F:H=√H:
    IF H>=GGOTO
    185
180:Z=H:H=G:G=Z:
    T=T+90
185:PRINT "BRG="
    ;J:PRINT "RN
    G=";K:GOTO 3
    20
200:"H":PRINT "T
    PE=A EST=Z S
    IZ=S PRB=X"
205:PRINT "CON=C
    ":GOTO 200
225:"S":INPUT "S
    IZE? ";S:O=1
    -EXP (-S*S/2
    )
230:PRINT USING
    "##.##";"PRB
    =";O:USING :
    GOTO 300
235:"X":INPUT "P
    RB? ";O:S=√(
    -2*LN (1-O))
    :PRINT "SIZE
    =";S
300:X=S*G:Y=S*H:
    N=T:PRINT "S
    MJ=";Y:IF N
    <=0LET N=N+1
    80
315:PRINT "DRC="
    ;N:PRINT "SM
    I=";X:Z=π*X*
    Y:PRINT "A="
    ;Z
320:PRINT "END":
    GOTO 320
325:PRINT "NO SO
    L":GOTO 325
400:X=U-R*SIN Q:
    Y=V-R*COS Q:
    GOSUB 500:W=
    (P-J):L=K*O*
    π/180:G=COS
    J/L:H=SIN J/
    L
415:IF W>=180LET
    W=W-360:GOTO
    440
425:IF W<=-180
    LET W=W+360
440:W=W/O:A=G*G+
    A:B=G*H+B:C=
    H*H+C:D=W*G+
    D:E=W*H+E:
    RETURN
500:K=√(X*X+Y*Y)
    :IF K=0LET J
    =0:RETURN
510:M=ASN (X/K):
    J=ACS (Y/K):
    IF M<0LET J=
    360-J
515:RETURN
```

```
5 INP "OPTION",J:SET F2:MODE 4
10 IF J=0 THEN 175
15 IF J=2 THEN 100
20 IF J=3 THEN 145
25 IF J=4 THEN 150
30 IF J=5 THEN 45
40 VAC :SET F2
45 INP "OBS BRG",P:INP "STA BRG",Q:INP "STA RNG",R
50 INP "BRG ERR",O:IF I=2 THEN 95
55 I=I+1:A(I-1)=P:A(I+1)=Q:A(I+3)=R:A(I+5)=O
60 IF I=1 THEN 45
65 X=A(4)*SIN (A(2)-A(0)):Y=A(5)*SIN (A(3)-A(1))
70 Z=SIN (A(1)-A(0)):IF Z=0 THEN 170
75 U=(X*SIN A(1)-Y*SIN A(0))/Z:V=(X*COS A(1)-Y*COS A(0))/Z
80 FOR M=0 TO 1:P=A(M):Q=A(M+2)
85 R=A(M+4):O=A(M+6):GSB 200:NEXT M:GOTO 45
90 R=A(M+4):O=A(M+6):GSB 200:NEXT M:GOTO 45
95 GSB 200:GOTO 45
100 F=(B*B-A*C):IF F=0 THEN 170
105 X=U+(B*E-C*D)/F:Y=V+(A*E-B*D)/F:RPC Y,X
110 T=SGN B*45:IF A=C THEN 120
115 T=.5*ATN (2*B/(A-C))
120 G=(C*COS T*COS T-2*B*COS T*SIN T+A*SIN T*SIN T)/-F
125 H=(C*SIN T*SIN T+2*B*COS T*SIN T+A*COS T*COS T)/-F
130 G=SQR G:H=SQR H:IF G>H;Z=H:H=G:G=Z:T=T+90
135 IF Y<0;Y=Y+360
140 PRT "BRG=";Y:PRT "RNG=";X:GOTO 165
145 INP "SIZE",S:O=1-EXP (-S*S/2):PRT "PRB=";O:GOTO 155
150 INP "PRB",O:S=SQR (-2*LN (1-O)):PRT "SIZE=";S
155 X=S*G:Y=S*H:PRT "SMJ=";Y:N=T:IF N<0;N=N+180
160 PRT "DIR=";N:PRT "SMI=";X:PRT "A=";π*X*Y
165 PRT "END":GOTO 165
170 PRT "NO SOL":GOTO 170
175 PRT "TPE=1 EST=2 SIZ=3"
180 PRT "PRB=4 CON=5":GOTO 175
200 X=U-R*SIN Q:Y=V-R*COS Q:RPC Y,X
205 W=(P-Y):L=X*O*π/180:G=COS Y/L:H=SIN Y/L
210 IF W≥180;W=W-360:GOTO 220
215 IF W≤-180;W=W+360
220 W=W/O:A=G*G+A:B=G*H+B:C=H*H+C:D=W*G+D:E=W*H+E:RET
```

000	47	CMS
001	60	DEG
002	02	2
003	42	STO
004	09	09
005	69	OP
006	28	28
007	76	LBL
008	16	A'
009	09	9
010	42	STO
011	00	00
012	42	STO
013	01	01
014	92	RTN
015	76	LBL
016	18	C'
017	69	OP
018	20	20
019	72	ST*
020	00	00
021	92	RTN
022	76	LBL
023	19	D'
024	69	OP
025	21	21
026	73	RC*
027	01	01
028	92	RTN
029	76	LBL
030	10	E'
031	65	×
032	89	π
033	55	÷
034	01	1
035	08	8
036	00	0
037	95	=
038	92	RTN
039	76	LBL
040	11	A
041	81	RST
042	76	LBL
043	12	B
044	18	C'
045	91	R/S
046	18	C'
047	91	R/S
048	18	C'
049	91	R/S
050	10	E'
051	18	C'
052	87	IFF
053	03	03
054	01	01
055	28	28
056	19	D'
057	75	-
058	19	D'
059	95	=
060	94	+/-
061	38	SIN
062	65	×
063	19	D'
064	95	=
065	48	EXC
066	19	19
067	22	INV
068	97	DSZ
069	09	09
070	00	00
071	76	76
072	01	1
073	69	OP
074	21	21
075	92	RTN
076	42	STO
077	18	18
078	65	×
079	43	RCL
080	14	14
081	42	STO
082	20	20
083	38	SIN
084	75	-
085	43	RCL
086	10	10
087	22	INV
088	44	SUM
089	20	20
090	38	SIN
091	65	×
092	43	RCL
093	19	19
094	95	=
095	55	÷
096	43	RCL
097	20	20
098	38	SIN
099	95	=
100	42	STO
101	38	38
102	43	RCL
103	14	14
104	39	COS
105	65	×
106	43	RCL
107	18	18
108	75	-
109	43	RCL
110	10	10
111	39	COS
112	65	×
113	43	RCL
114	19	19
115	95	=
116	55	÷
117	43	RCL
118	20	20
119	38	SIN
120	95	=
121	42	STO
122	39	39
123	86	STF
124	03	03
125	02	2
126	42	STO
127	09	09
128	16	A'
129	19	D'
130	10	E'
131	42	STO
132	18	18
133	19	D'
134	32	X:T
135	19	D'
136	32	X:T
137	37	P/R
138	75	-
139	43	RCL
140	38	38
141	95	=
142	94	+/-
143	32	X:T
144	75	-
145	43	RCL
146	39	39
147	95	=
148	94	+/-
149	32	X:T

150	22	INV
151	37	P/R
152	42	STO
153	19	19
154	10	E'
155	75	-
156	43	RCL
157	18	18
158	95	=
159	94	+/-
160	32	X:T
161	69	OP
162	21	21
163	64	PD*
164	01	01
165	42	STO
166	18	18
167	89	π
168	32	X:T
169	22	INV
170	77	GE
171	01	01
172	78	78
173	75	-
174	32	X:T
175	65	×
176	02	2
177	95	=
178	49	PRD
179	18	18
180	73	RC*
181	01	01
182	35	1/X
183	42	STO
184	27	27
185	42	STO
186	26	26
187	32	X:T
188	43	RCL
189	19	19
190	37	P/R
191	42	STO
192	28	28
193	49	PRD
194	27	27
195	33	X²
196	44	SUM
197	23	23
198	43	RCL
199	18	18
200	49	PRD
201	26	26
202	49	PRD
203	27	27
204	32	X:T
205	49	PRD
206	28	28
207	49	PRD
208	26	26
209	33	X²
210	44	SUM
211	21	21
212	43	RCL
213	28	28
214	44	SUM
215	22	22
216	43	RCL
217	26	26
218	44	SUM
219	24	24
220	43	RCL
221	27	27
222	44	SUM
223	25	25
224	22	INV
225	97	DSZ
226	09	09
227	02	02
228	32	32
229	61	GTO
230	01	01
231	29	29
232	69	OP
233	28	28
234	43	RCL
235	08	08
236	92	RTN
237	76	LBL
238	13	C
239	43	RCL
240	21	21
241	42	STO
242	41	41
243	43	RCL
244	22	22
245	42	STO
246	42	42
247	43	RCL
248	23	23
249	42	STO
250	43	43
251	42	STO
252	10	10
253	65	×
254	43	RCL
255	21	21
256	22	INV
257	44	SUM
258	10	10
259	75	-
260	43	RCL
261	22	22
262	22	INV
263	49	PRD
264	10	10
265	33	X²
266	95	=
267	35	1/X
268	49	PRD
269	41	41
270	49	PRD
271	42	42
272	49	PRD
273	43	43
274	02	2
275	55	÷
276	43	RCL
277	10	10
278	95	=
279	24	CE
280	22	INV
281	30	TAN
282	55	÷
283	02	2
284	95	=
285	42	STO
286	29	29
287	43	RCL
288	43	43
289	65	×
290	43	RCL
291	24	24
292	75	-
293	43	RCL
294	42	42
295	65	×
296	43	RCL
297	25	25
298	85	+
299	43	RCL

300	38	38
301	95	=
302	32	X⇌T
303	43	RCL
304	42	42
305	65	×
306	43	RCL
307	24	24
308	75	-
309	43	RCL
310	41	41
311	65	×
312	43	RCL
313	25	25
314	85	+
315	43	RCL
316	39	39
317	95	=
318	32	X⇌T
319	22	INV
320	37	P/R
321	42	STO
322	32	32
323	01	1
324	32	X⇌T
325	42	STO
326	33	33
327	43	RCL
328	29	29
329	37	P/R
330	42	STO
331	11	11
332	33	X²
333	42	STO
334	12	12
335	42	STO
336	13	13
337	32	X⇌T
338	49	PRD
339	11	11
340	33	X²
341	42	STO
342	14	14
343	42	STO
344	15	15
345	43	RCL
346	41	41
347	49	PRD
348	12	12
349	49	PRD
350	14	14
351	43	RCL
352	42	42
353	65	×
354	02	2
355	95	=
356	49	PRD
357	11	11
358	43	RCL
359	43	43
360	49	PRD
361	13	13
362	49	PRD
363	15	15
364	43	RCL
365	15	15
366	85	+
367	43	RCL
368	11	11
369	85	+
370	43	RCL
371	12	12
372	95	=
373	34	√X
374	42	STO
375	16	16
376	43	RCL
377	13	13
378	75	-
379	43	RCL
380	11	11
381	85	+
382	43	RCL
383	14	14
384	95	=
385	34	√X
386	42	STO
387	17	17
388	43	RCL
389	32	32
390	91	R/S
391	43	RCL
392	33	33
393	92	RTN
394	61	GTO
395	03	03
396	93	93
397	76	LBL
398	15	E
399	75	-
400	01	1
401	95	=
402	94	+/-
403	23	LNX
404	65	×
405	02	2
406	95	=
407	94	+/-
408	34	√X
409	91	R/S
410	42	STO
411	15	15
412	65	×
413	43	RCL
414	16	16
415	95	=
416	42	STO
417	18	18
418	32	X⇌T
419	43	RCL
420	29	29
421	94	+/-
422	42	STO
423	30	30
424	43	RCL
425	15	15
426	65	×
427	43	RCL
428	17	17
429	95	=
430	77	GE
431	04	04
432	41	41
433	48	EXC
434	18	18
435	32	X⇌T
436	09	9
437	00	0
438	44	SUM
439	30	30
440	32	X⇌T
441	91	R/S
442	32	X⇌T
443	43	RCL
444	30	30
445	91	R/S
446	43	RCL
447	18	18
448	91	R/S
449	65	×

```
450  32 X:T
451  65  ×
452  89  π
453  95  =
454  92 RTN
455  61 GTO
456  04  04
457  54  54
458  76 LBL
459  14  D
460  33 X²
461  55  ÷
462  32 X:T
463  02  2
464  95  =
465  94 +/-
466  22 INV
467  23 LNX
468  75  -
469  01  1
470  95  =
471  94 +/-
472  91 R/S
473  32 X:T
474  34 √X
475  61 GTO
476  04  04
477  10  10
478  00  0
479  00  0
```

Appendix 1. The Estimation Procedure

In this development the assumptions stated in Section I are required conditions. A rectangular coordinate system is used with the positive y-axis directed north, the positive x-axis directed east and the origin at the reference point and all angles are in radians. Figure 3 shows a station located with respect to the coordinate system. The bearing line of length r goes to the object's unknown position, the bearing line of length r goes to an initial estimate of the object's position and the third bearing line corresponds to an observed bearing.

To determine the coordinates for a final estimate, consider the arc segments $u = r(\theta-\phi)$ where $\theta-\phi$ is the bearing error and $v = r(\phi-\beta)$ and $w = r(\theta-\beta)$ that are defined by the three bearing lines and the circle of radius r that is centered on the station and goes through the initial estimate. The geometry is shown in Figure 3. Note that u can be defined by $u = w - v$. In this expression, $w = r(\theta-\beta)$ is known, and v can be determined in terms of x and y the unknown coordinates of the target. With the reference point at the initial estimte, to first order, $v = x \cos \beta - y \sin \beta$ and $u = w - x \cos \beta + y \sin \beta$. Since this approximation applies to all stations, its use suggests that, for each station, $r \simeq r$ which is equivalent to having the initial estimate relatively close to the target's position.

Since, for each station i, an observed bearing θ_i is the value of a normal random variable Θ_i with mean ϕ_i, the coordinate $u_i = r_i(\theta_i-\phi_i)$ is the value of a normal random variable U_i with mean zero. In addition, since the Θ_i are independent, the U_i are

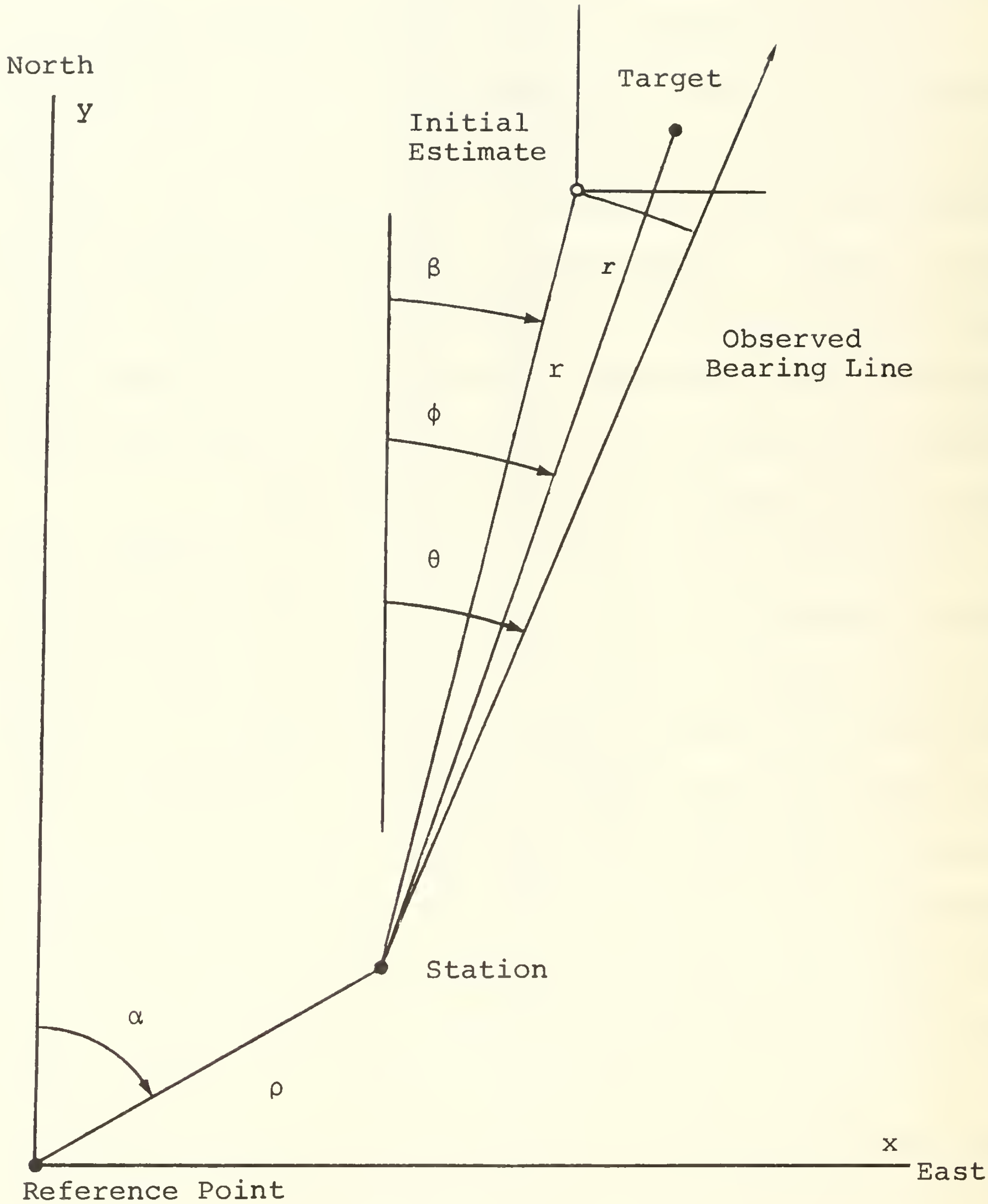

Figure 3. The coordinate geometry. The coordinates of the initial estimate are (x^*,y^*). In the development, the reference point is at the initial estimate.

also independent. And, consequently, the joint distribution of the U_i is determined.

To estimate x and y, maximum likelihood estimates are used here. The likelihood for a sample θ_i, $\theta_2, \ldots, \theta_n$ from n stations is

$$L(\theta_1, \theta_2, \ldots, \theta_n) = \prod_1^n \frac{1}{\sqrt{2\pi}\, e_i} \exp - \frac{1}{2} \sum_1^n (\theta_i - \phi_i)^2 / e_i^2$$

and the likelihood for a cooresponding sample u_1, $u_2, \ldots, u_n$ is

$$L(u_1, u_2, \ldots, u_n) = \prod_1^n \frac{1}{\sqrt{2\pi}\, \sigma_i} \exp - \frac{1}{2} \sum_1^n u_i^2 / \sigma_i^2$$

with $\sigma_i = r_i e_i$ where e_i is the standard deviation of Θ_i.

By definition, the maximum likelihood estimates for x and y are the estimates which make $L(u_1, u_2, \ldots, u_n)$ a maximum. In this case, making $L(u_1, u_2, \ldots, u_n)$ a maximum is equivalent to making $\sum_1^n (u_i^2/\sigma_i^2)$ a minimum. So, to find the maximum likelihood estimates $\hat{x}$ and $\hat{y}$, solve the following two equations for $\hat{x}$ and $\hat{y}$:

$$\left.\frac{\partial(\ln L)}{\partial x}\right|_{x=\hat{x}} = 0 \quad \text{and} \quad \left.\frac{\partial(\ln L)}{\partial y}\right|_{y=\hat{y}} = 0$$

With $u_i = w_i - x \cos \beta_i + y \sin \beta_i$, the equations can be written as follows:

$$\sum_1^n (w_i - \hat{x} \cos \beta_i + \hat{y} \sin \beta_i)(\cos \beta_i)/\sigma_i^2 = 0$$

and

$$\sum_1^n (w_i - \hat{x} \cos \beta_i + \hat{y} \sin \beta_i)(\sin \beta_i)/\sigma_i^2 = 0 .$$

In terms of the following quantities:

$$A = \Sigma(\cos^2 \beta_i)/\sigma_i^2 , \qquad B = \Sigma(\sin \beta_i \cos \beta_i)/\sigma_i^2 ,$$

$$C = \Sigma(\sin^2 \beta_i)/\sigma_i^2 , \qquad D = \Sigma(w_i \cos \beta_i)/\sigma_i^2 ,$$

$$E = \Sigma(w_i \sin \beta_i)/\sigma_i^2 ,$$

the equations become:

$$A\hat{x} - B\hat{y} = D$$

$$B\hat{x} - C\hat{y} = E$$

The solutions are:

$$\hat{x} = (BE - CD)/(B^2 - AC) \tag{1}$$

and

$$\hat{y} = (AE - BD)/(B^2 - AC) \tag{2}$$

A confidence region can be constructed about an estimated position. In order to indicate how this can be done, a probability region about the true position will be considered first.

Both $\hat{x}$ and $\hat{y}$ are values of random variables. If a new set of bearings $\theta_1, \theta_2, \ldots, \theta_n$ is observed (for the same initial estimate and a fixed target), in general, a new pair of values $\hat{x}$ and $\hat{y}$ will be obtained.

If $\hat{X}$ and $\hat{Y}$ represent these random variables,

$$\hat{X} = \frac{1}{(B^2-AC)} \sum_1^n (W_i/\sigma_i^2)(B \sin \beta_i - C \cos \beta_i)$$

$$\hat{Y} = \frac{1}{(B^2-AC)} \sum_1^n (W_i/\sigma_i^2)(A \sin \beta_i - B \cos \beta_i)$$

where $W_i = r_i(\Theta_i - \beta_i)$.

Since $\hat{X}$ and $\hat{Y}$ are a linear combination of the n normal random variables $W_1, W_2, \ldots, W_n$, or equivalently of the n normal random variables $\Theta_1, \Theta_2, \ldots, \Theta_n$, they have a joint normal distribution. Since $E(W_i) = r_i(\phi_i - \beta_i)$, if $\beta_i = \phi_i$ for i = 1, 2, ..., n, that is, if the initial estimate of the target's position is at the target's position, $E(W_1) = 0$ for i = 1, 2, ..., n. In this case $E(\hat{X}) = 0$ and $E(\hat{Y}) = 0$ and the joint normal distribution is centered on the object's position. To the degree of the approximations that have been made, this is also true if the initial estimate is not at the target's position.

A region of minimum area for a given probability of containment of an estimated position can be determined. The region is bounded by an ellipse which is centered on the object's position and whose axes lie along the axes of an x'y'-coordinate system that is obtained by rotating the xy-coordinate system that is centered on the object's position through an angle γ. In this system, $\sigma_{\hat{x}'\hat{y}'}$ is 0, that is $\hat{x}'$ and $\hat{y}'$ are independent normal random variables. The two coordinate systems are illustrated in

Figure 4. The coordinates of a point in the two systems are related by

$$x' = x \cos \gamma - y \sin \gamma$$

$$y' = x \sin \gamma + y \cos \gamma$$

These relations imply:

$$(3) \quad \sigma^2_{\hat{x}'} = \sigma^2_{\hat{x}} \cos^2 \gamma - 2\sigma_{\hat{x}\hat{y}} \cos \gamma \sin \gamma + \sigma^2_{\hat{y}} \sin^2 \gamma \; ,$$

$$(4) \quad \sigma^2_{\hat{y}'} = \sigma^2_{\hat{x}} \sin^2 \gamma + 2\sigma_{\hat{x}\hat{y}} \cos \gamma \sin \gamma + \sigma^2_{\hat{y}} \cos^2 \gamma$$

and

$$(5) \quad \sigma^2_{\hat{x}'\hat{y}'} = (\sigma^2_{\hat{x}} - \sigma^2_{\hat{y}}) \sin \gamma \cos \gamma + \sigma_{\hat{x}\hat{y}} (\cos^2 \gamma - \sin^2 \gamma)$$

where γ, the angle of rotation of the coordinate axes, is positive in the clockwise direction. And $\sigma_{\hat{x}'\hat{y}'} = 0$ implies

$$\tan 2\gamma = \frac{2\sigma_{\hat{x}\hat{y}}}{\sigma^2_{\hat{y}} - \sigma^2_{\hat{x}}}$$

With the initial estimate of the target's position at the target's position $E(W_i) = 0$ and therefore, $\text{Var}(W_i) = \sigma_i{}^2$ for $i = 1, 2, \ldots, n$. In this case

$$\sigma^2_{\hat{x}} = \frac{1}{(B^2 - AC)^2} \sum_{1}^{n} (1/\sigma_i^2)(B \sin \beta_i - C \cos \beta_i)^2,$$

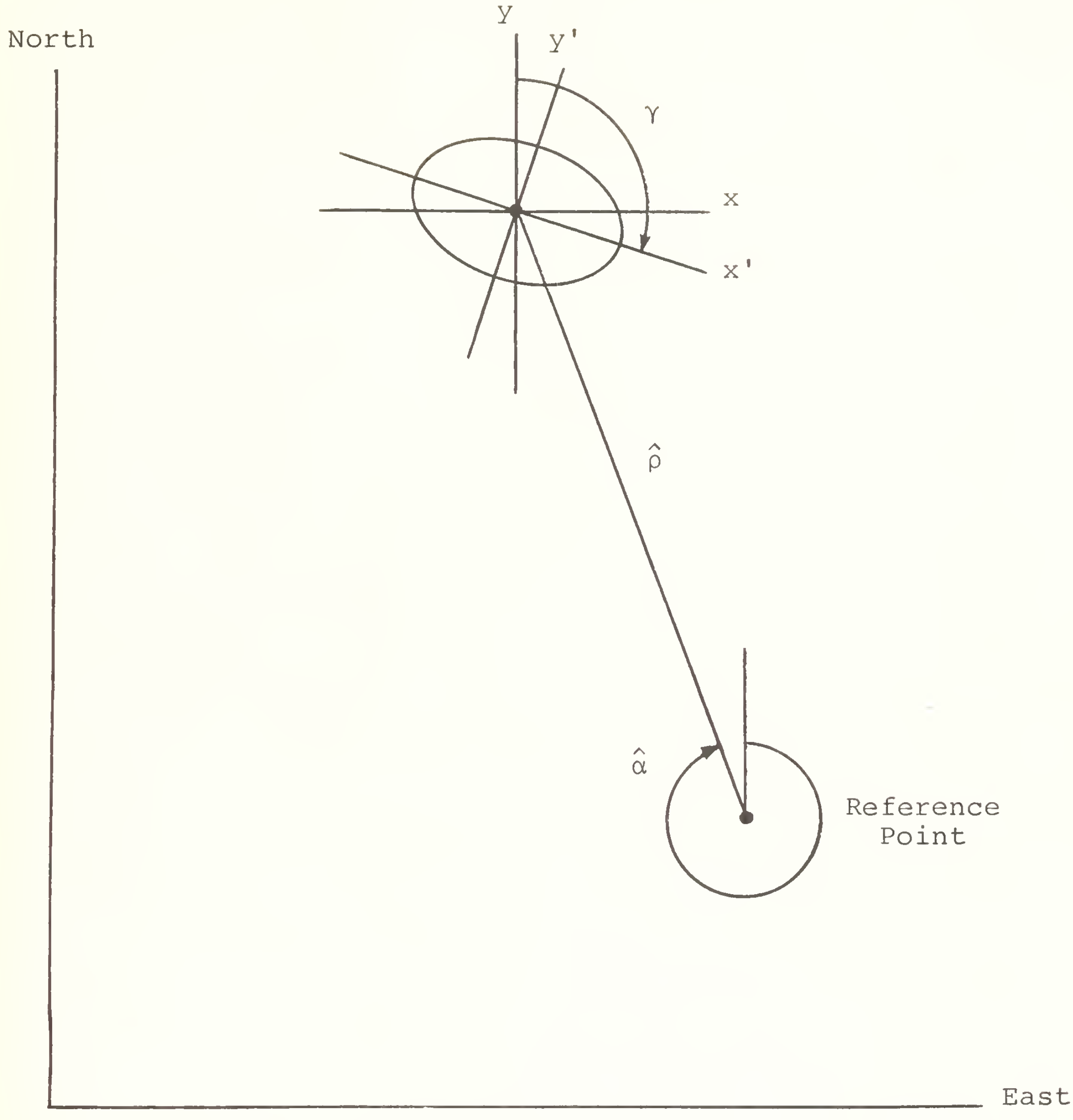

Figure 4. An elliptical confidence region and the primed coordinate system in which the covariance $\sigma_{\hat{x}',\hat{y}'}$ is zero. The center of the ellipse and the origin of the coordinate systems are at the target's estimated position. The estimated bearing $\hat{\alpha}$ and estimated range $\hat{\rho}$ are indicated for a reference position.

$$\sigma^2_{\hat{y}} = \frac{1}{(B^2-AC)^2} \sum_1^n (1/\sigma_i^2)(A \sin \beta_i - B \cos \beta_i)^2$$

and

$$\sigma_{\hat{x}\hat{y}} = \frac{1}{(B^2-AC)} \sum_1^n (1/\sigma_i^2)(B \sin \beta_i - C \cos \beta_i)(A \sin \beta_i - B \cos \beta_i).$$

Using the definition for A, B and C, the above become

(6) $$\sigma^2_{\hat{x}} = \frac{C}{(AC-B^2)},$$

(7) $$\sigma^2_{\hat{y}} = \frac{A}{(AC-B^2)},$$

and

(8) $$\sigma_{\hat{x}\hat{y}} = \frac{B}{(AC-B^2)}.$$

So tan 2 = 2B/(A-C) for $\beta_i = \phi_i$, i = 1, 2, ..., n.

With the target's position known and, consequently, ϕ_i known for i = 1, 2, ..., n, the above expressions for $\sigma^2_{\hat{x}}$, $\sigma^2_{\hat{y}}$, $\sigma_{\hat{x}\hat{y}}$ and γ can be used, since the initial estimate of the target's position can be taken as the target's position.

With values for $\sigma_{\hat{x}}$, $\sigma_{\hat{y}}$, $\sigma_{\hat{x}\hat{y}}$ and γ, values for $\sigma_{\hat{x}'}$ and $\sigma_{\hat{y}'}$ can be found by using equations (3) and (4). The probability that an estimated position will be within an ellipse of semiaxes $k\sigma_{\hat{x}'}$ and $k\sigma_{\hat{y}'}$ which is centered on the target's position is

$1 - \exp(-k^2/2)$. This result can be found by integrating the bivariate normal density over the ellipse. And the area of the ellipse is $k^2 \sigma_{\hat{x}'} \sigma_{\hat{y}'}$.

Given estimates $\hat{x}$ and $\hat{y}$ found by using Equations (1) and (2), the ellipse with semi-axes $k\sigma_{\hat{x}'}$ and $k\sigma_{\hat{y}'}$ in a coordinate system that is centered on the point $(\hat{x},\hat{y})$ and has been rotated through an angle γ is a $1 - \exp(-k^2/2)$ confidence region. This follows, since, to the degree of the approximations involved, the bivariate normal distribution of X and Y is centered on the target's position. The confidence ellipse is defined if $\sigma^2_{\hat{x}}$, $\sigma^2_{\hat{y}}$ and $\sigma_{\hat{x}\hat{y}}$ can be found, that is if the elements of the covariance matrix can be found. To the degree of the approximations involved, this can be done as follows: First, assume the initial estimate of the target's position is at the target's position. Then, values for $\sigma^2_{\hat{x}}$, $\sigma^2_{\hat{y}}$, $\sigma_{\hat{x}\hat{y}}$ and γ can be determined by using Equations (6), (7) and (8). These values can then be used to determine $\sigma^2_{\hat{x}'}$, $\sigma^2_{\hat{y}'}$ and $\sigma_{\hat{x}'\hat{y}'}$ by using Equations (3), (4) and (5). Now, with a value for k, a confidence region can be constructed. To the degree of the approximations involved, the shape of the confidence region is independent of both the target's position and of the initial estimate of the target's position.

For the case where bearings are taken from the target on two or more stations, θ_i is the reciprocal of the bearing taken from the target.

A discussion of the theory of bearings only position estimation procedures for situations similar to the one considered here is given in Reference 1. Reference 2 gives an essentially

equivalent bearings only procedure. It also gives a range only procedure, a range and bearing procedure and HP-9830A programs with which to implement the procedures. Using the fix determined by two lines of bearing as the initial estimate was suggested by this reference.

The following equations are evaluated in the program to determine the coordinates of the initial estimate:

$$x^* \sin (\theta_2-\theta_1) = [\rho_1 \sin (\alpha_1-\theta_1)] \sin \theta_2$$

$$- [\rho_2 \sin (\alpha_2-\theta_2)] \sin \theta_1$$

and

$$y^* \sin (\theta_2-\theta_1) = [\rho_1 \sin (\alpha_1-\theta_1)] \cos \theta_2$$

$$- [\rho_2 \sin (\alpha_2-\theta_2)] \cos \theta_1 \ .$$

Reference 3 describes a TI-59 program that is based on an equivalent procedure. The program allows a user to either input the coordinates of the initial estimate or determine them in the manner described here.

Appendix 2. HP-41CV Program Labels

The global labels in the HP-41CV program that are assigned to the keys $\Sigma+$, 1/x, $\sqrt{x}$, LOG and LN give these keys a mnemonic character. For example, with the calculator in USER mode, if LOG is pressed and held, SIZ will be displayed and then after a delay, NULL.

If a global label is replaced by a local label, this mnemonic character will be lost. However, automatic key assignment will be gained if the key corresponding to the local label has not been previously assigned. If this is the case and if there is a second program in program memory that uses the same local label, then it will be automatically assigned when the calculator is positioned in program memory at that program.

Global labels in the HP-41CV program can be replaced with local labels. If this is done as described below, the user instructions will still be applicable. First, make the following replacements:

Line 329: LBL "PRB" → LBL E. Line 311: LBL "SIZ" → LBL D.

Line 186: LBL "EST" → LBL B. Line 008: LBL "CON" → LBL C.

Line 047: GTO "CON" → GTO C.

Next, after line 001: LBL "TPE", insert LBL A so that it becomes line 002. Finally, if there is a key assignment for a key in the following list: $\Sigma+$, 1/x, $\sqrt{x}$, LOG and LN, remove it. Now, with the calculator in USER mode, if LOG is pressed and held, XEQ D will be displayed and then, after a delay, NULL. The key's mnemonic character has been lost, but if there is a second program in program memory with the local label D, LOG will be automatically assigned to D when the calculator is positioned in program memory at that program.

REFERENCES

1. Daniels, H. E., "The Theory of Position Finding," J. Royal Stat. Soc. (B), Vol. 13, 1951, pp. 186-207.

2. Thompson, K. P. and Kullback, J. H., "Position-Fixing and Position-Predicting Programs for the Hewlett-Packard Model 9830A Programmable Calculator," NRL Memorandum Report 3265, Naval Research Laboratory, Washington, DC , April 1976.

3. Forrest, R. N., "A Procedure for Estimating an Object's Position Based on Two or More Bearings with a Program for a TI-59 Calculator," NPS55-77-34 (Revised), Naval Postgraduate School, Monterey, CA., 93940, September 1977 (Revised August 1978).

INITIAL DISTRIBUTION LIST

	COPIES
Dean of Research Code 012 Naval Postgraduate School Monterey, California 93940	1
Defense Technical Information Center Cameron Station Alexandria, Virginia 22314	2
Library, Code 0212 Naval Postgraduate School Monterey, California 93940	2
Office of Naval Research Attn: Code ONR-230 800 North Quincy Street Arlington, Virginia 22217	3
Commander Submarine Development Squadron 12 Naval Submarine Base, New London Groton, Connecticut 06340	1
Commander Surface Warfare Development Group Naval Amphibious Base, Little Creek Norfolk, Virginia 23521	1
Director Strategic Systems Projects Office Attn: Code SP2024, Code 20231 1931 Jefferson Davis Highway Arlington, Virginia 20376	2
Navy Tactical Support Activity Attn: M. E. Lannon P. O. Box 1042 Silver Spring, Maryland 20910	1
Naval Air Development Center Johnsville, Pennsylvania 18974	1
Naval Surface Weapons Center White Oak Silver Spring, Maryland 20910	1
Naval Underwater Systems Center Newport, Rhode Island 02840	1
Naval Underwater Systems Center New London, Connecticut 06320	1

Naval Ocean Systems Center San Diego, California 92132	1
Naval Intelligence Support Center Attn: Code NISC 20, Library 4301 Suitland Road Washington, DC 20390	2
Naval Electronic Systems Command 2511 Jefferson Davis Highway Arlington, Virginia 20360	1
Naval Air Systems Command Code 370 Washington, DC 20361	1
Antisubmarine Warfare Systems Project Office Department of the Navy Washington, DC 20360	1
Center for Naval Analysis 1401 Wilson Boulevard Arlington, Virginia 22209	1
Code 55Fo Naval Postgraduate School Monterey, California 93940	200